CATEGORY THEORY
TOPIC 22: METRIC SPACES

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1. Prerequisites

We will adopt the convention that complements are assumed to be with respect to some predetermined fixed set X. Thus, for $A \subset X$, define

$$A^c = X \smallsetminus A = \{z \in X \mid z \notin A.\}$$

We will need *DeMorgan's Laws* of set theory, which state:

- $X \smallsetminus (A \cup B) = (X \smallsetminus A) \cap (X \smallsetminus B)$
- $X \smallsetminus (A \cap B) = (X \smallsetminus A) \cup (X \smallsetminus B)$

Using complements in X, this becomes

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

We may rephrase these in words:

- If z is not in either A or B, then z is not in A and z is not in B.
- If z is not in both A and B, then z is not in A or z is not in B.

It should be noted that DeMorgan's Laws may be generalized to unions and intersections of arbitrarily many sets.

By common convention, if we say, "let $r > 0$", we mean "let r denote a positive real number".

If A and B are sets, we say that A *intersects* B is $A \cap B \neq \varnothing$.

Date: January 20, 2020.

2. Metric Spaces

Definition 1. Let X be a set. A *metric* on X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying

(M1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (Positivity);
(M2) $d(x, y) = d(y, x)$ (Symmetry);
(M3) $d(x, y) + d(y, z) \geq d(x, z)$ (Triangle Inequality).

The pair (X, d) is called a *metric space*.

Example 1. The set of real numbers is a metric space. The distance from x to y is defined by $d(x, y) = |x - y|$.

Example 2. Let $X = \mathbb{R}^2$ and use the Pythagorean theorem to define the metric d by

$$d(p, q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

where $p = (x_1, y_1)$ and $q = (x_2, y_2)$.

Example 3. Let $X = \mathbb{R}^3$. Two applications of the Pythagorean theorem and some slight simplification leads to the definition of the metric d by

$$d(p, q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

where $p = (x_1, y_1, z_1)$ and $q = (x_2, y_2, z_2)$.

Example 4. Let $X = \mathbb{R}^n$. We need to slightly modify our notation to conveniently write the distance formula. Thus for $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$, define

$$d(p, q) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Example 5. Let $\mathbb{R}^\infty$ denote the set of all sequences of real numbers that are eventually zero, that is, sequences (x_n) such that $x_n = 0$ for all but finitely many n. Let $X = \mathbb{R}^\infty$ and for $x, y \in X$, define

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $x = (x_n)$ and $y = (y_n)$. This make sense, since there are only finitely many nonzero summands. Then (X, d) is a metric space.

Example 6. Let X be any set and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise }. \end{cases}$$

Then d is a metric on X, called the *discrete metric*, and (X, d) is called a *discrete metric space*.

Example 7. Let $\mathcal{F}_{[a,b]}$ denote the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{F}_{[a,b]}$ and for $f, g \in X$ define

$$d(f, g) = \max\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

Then (X, d) is a metric space.

3. Open Sets

Let (X, d) be a metric space. We make the following definitions.

Definition 2. Let $a \in X$ and $r > 0$. The *ball of radius r about a* is the set

$$B_r(a) = \{z \in X \mid d(a, z) < r\}.$$

Definition 3. Let $U \subset X$. We say that U is *open* if, for every $u \in U$ there exists $r > 0$ such that $B_r(u) \subset U$.

Proposition 1. *Let $a \in X$ and let $r > 0$. Then $B_r(a)$ is open.*

Proposition 2. *Let $U, V \subset X$ be open sets. Then $U \cap V$ is open.*

Proposition 3. *A subset of X is open if and only if it is a union of open balls.*

Reason. Let $U \subset X$. If U is open, then every point in U admits an open ball that is contained in U. Select such an open ball for each point in U, and take the union of these balls, and you will obtain U. On the other hand, suppose U is a union of open balls, and let $u \in U$. Then u is an element of some open ball, say $u \in B_r(v)$ for some $v \in U$. Let $s = r - d(u, v)$; then $B_s(v) \subset U$. $\square$

Proposition 4. *The following are true.*

- **(T1)** *$\varnothing$ and X are open sets.*
- **(T2)** *The union of any number of open sets is an open set.*
- **(T3)** *The intersection of finitely many open sets if an open set.*

Reason. The empty set is vacuous open, since we cannot find a point in the empty set which violates the condition for openness. For every $u \in X$, we have $B_1(u) \subset X$, so X is open.

Each open set is the union of open balls, so the union of a collection of open sets is also a union of open balls, and so is also open.

Since the intersection of two open sets is open by Proposition 2, induction implies that the intersection of finitely many open sets is also open. $\square$

It is not the case that the intersection of infinitely many open sets is necessarily open. To see this, consider that the intersection of all open balls around a given point is the singleton set containing that point; for example,

$$\bigcap_{n \in \mathbb{N}} B_{1/n}(0) = \{0\}.$$

4. Closed Sets

Definition 4. Let $F \subset X$. We say that F is *closed* if its complement, F^c, is open.

We may apply DeMorgan's Laws to Proposition 4 to obtain the following.

Proposition 5. *The following are true.*

(F1) $\varnothing$ *and* X *are closed sets.*
(F2) *The intersection of any number of closed sets is a closed set.*
(F3) *The union of finitely many closed sets is a closed set.*

5. Neighborhoods

Definition 5. Let $a \in X$ A *neighborhood* of a is a subset $N \subset X$ such that there exists an open set $U \subset N$ with $a \in U$.

Remark 1. Let $a \in X$. It is immediate that if N is a neighborhood of a, then $a \in N$.

If U is an open set containing a, then U is itself a neighborhood of a, and is referred to as an open neighborhood. Thus there exists at least one neighborhood of a; indeed, X is open and contains a.

We are interested in sets A whose intersection with neighborhoods of a are nonempty, in which case we say that A intersects the neighborhood. If $a \in A$, then every neighborhood of a intersects A; this is the less interesting case for us.

Clearly A intersects every neighborhood of a if and only if A intersects every open neighborhood of a; the forward direction is immediate and the reverse direction is given by considering a neighborhood which does not intersect A, which must contain an open neighborhood which does not intersect A.

Definition 6. Let $a \in X$. A *deleted neighborhood* of a is a set of the form $N \smallsetminus \{a\}$, where N is a neighborhood of a.

Remark 2. Let $a \in X$. If $N \smallsetminus \{a\}$ is a deleted neighborhood of a which does not intersect A, then either N does not intersect A or there is an open set $U \subset N$ such that a is the only element of A in that open set.

6. Classification of Points

6.1. Interior Points.

Definition 7. Let $A \subset X$. An *interior point* of A is a point $z \in A$ such that A contains a neighborhood of z. The *interior* of A is the set of interior points of A and is denoted A°.

Proposition 6. *Let $A \subset X$. Then $A^\circ \subset A$.*

Proof. Let $a \in A^\circ$. Then there exists a neighborhood of a which is contained in A. Since a is in this neighborhood, it is the case that $a \in A$. So $A^\circ \subset A$. □

Proposition 7. *Let $A \subset X$. Then A° open.*

Proof. Let $a \in A^\circ$. Then a is an interior point of A, so there exists an open neighborhood U_a of a such that $U_a \subset A$. If $u \in U_a$, then U_a is also an open neighborhood of u, so that $u \in A^\circ$; thus $U_a \subset A^\circ$.

Thus for each $a \in A^\circ$, let U_a be a open neighborhood of a which is contained in A°. Let $U = \bigcup_{a \in A} U_a$; since U is a union of open sets, U is open.

We claim that $A^\circ = U$. To see this, consider that if $a \in A^\circ$, then $a \in INTU_a$, so $a \in U$, so $A^\circ \subset U$. On the other and, if $u \in U$, then $u \in U_a$ for some a, and $U_a \subset A^\circ$, so $u \in A^\circ$, so $U \subset A^\circ$. Thus $U = A^\circ$, and since U is a union of open sets, A° is open. □

Proposition 8. *Let $A \subset X$. Then A is open if and only if $A = A^\circ$.*

Proof. Suppose A is open. We already know that $A^\circ \subset A$. Since A is a neighborhood of every point in A, every point in A is an interior point, so $A \subset A^\circ$.

Suppose that $A = A^\circ$. Since A° is open, so is A. □

Proposition 9. *Let $A \subset X$. Then A is open if and only if every point in A is an interior point.*

Proof. This is just a rewording of the previous proposition. □

Proposition 10. *Let $A \subset X$. The interior of A is the union of all open sets which are contained in A.*

Proof. Let $\mathfrak{U} = \{U \subset X \mid U \text{ is open and } U \subset A\}$. We wish to show that $A^\circ = \cup\mathfrak{U}$.

Since A° is an open set which is contained in A, $A^\circ \in \mathfrak{U}$, so $A^\circ \subset \cup\mathfrak{U}$.

On the other hand, every point $u \in \mathfrak{U}$ is in U for some $U \in \mathfrak{U}$, so U is a neighborhood of u which is contained in A, so u is an interior point of A, ad $u \in A^\circ$; thus $\cup\mathfrak{U} \subset A^\circ$. □

Proposition 11. *Let $A, B \subset X$. Then*

(a) $A \subset B \Rightarrow A^\circ \subset B^\circ$;
(b) $(A^\circ)^\circ = A^\circ$;
(c) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof. Exercise. □

6.2. **Closure Points.**

Definition 8. Let $A \subset X$. A *closure point* of A is a point $z \in X$ such that every neighborhood of z intersects A. The *closure* of $A \subset X$ is the set of closure points of A and is denoted $\overline{A}$.

Proposition 12. *Let $A \subset X$. Then $A \subset \overline{A}$.*

Proof. Let $a \in A$. Every neighborhood of a contains a, and since $a \in A$, every neighborhood of a intersects A. Thus a is a point of closure of A, so $a \in \overline{A}$. $\square$

Proposition 13. *Let $A \subset X$. Then $\overline{A}$ is closed.*

Proof. Wish to show that $\overline{A}^c$ is open. Thus let $u \in \overline{A}^c$, so that u is not a closure point of A. This means that there exists an open neighborhood U of u which does not intersect A. If $v \in U$, then U is a neighborhood of v which does not intersect A, so v is not a closure point of A. Hence $U \subset \overline{A}^c$, which shows that u is an interior point of $\overline{A}^c$, so $\overline{A}^c$ is open. Therefore $\overline{A}$ is closed. $\square$

Proposition 14. *Let $A \subset X$. Then A is closed if and only if $A = \overline{A}$.*

Proof. Suppose A is closed. We have already seen that $A \subset \overline{A}$. Now suppose that $u \notin A$. Then $u \in A^c$, which is open, so there exists a neighborhood U of u which is contained in A^c, so U does not intersect A. Thus u is not a closure point of A, so $u \notin \overline{A}$. Thus $a \in A$ if and only if $a \in \overline{A}$, so $A = \overline{A}$.

On the other hand, if $A = \overline{A}$, then A is closed, since $\overline{A}$ is closed. $\square$

Proposition 15. *Let $A \subset X$. Then A is closed if and only if every point in A is an closure point.*

Proof. This is just a rewording of the previous proposition. $\square$

Proposition 16. *Let $A \subset X$. Then $\overline{A}$ is the intersection of the closed subsets of X which contain A.*

Proof. Let $\mathcal{F} = \{F \subset X \mid F \text{ is closed and } A \subset F\}$. We wish to show that $\overline{A} = \cap\mathcal{F}$.

Since $\overline{A}$ is a closed set which contains A, $\overline{A} \in \mathcal{F}$, so $\cap\mathcal{F} \subset \overline{A}$.

Now suppose $u \notin \cap\mathcal{F}$. Then $u \notin F$ for some $F \in \mathcal{F}$. Thus $u \in F^c$, which is open, so there exists an open neighborhood U of u which is contained in $F^c \subset X \setminus A$. That is, U does not intersect A, and u is not a closure point of A. Thus $b \in \overline{A}$ if and only if $b \in \cap\mathcal{F}$, so $\overline{A} = \cap\mathcal{F}$. $\square$

Proposition 17. (Kuratowski Closure Operator)
The following are true.

(K1) $\overline{\varnothing} = \varnothing$;
(K2) $A \subset \overline{A}$;
(K3) $\overline{\overline{A}} = \overline{A}$;
(K4) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

Proof. The first two are immediate from the definition.

From **(K2)** we have $\overline{A} \subset \overline{\overline{A}}$. Suppose that $x \in \overline{\overline{A}}$. Then every open neighborhood of x intersects $\overline{A}$. For any open neighborhood U of x, let $y \in U \cap \overline{A}$. Then every open neighborhood of y intersects A. Since U is an open neighborhood of y, U intersects A. Thus $x \in \overline{A}$.

Suppose that $x \notin \overline{A} \cup \overline{B}$. Then there exists a neighborhoods U, V of x such that $U \cap A = \varnothing$ and $V \cap B = \varnothing$. Then $U \cap V$ is a neighborhood of x such that $(U \cap V) \cap (A \cup B) = \varnothing$. So $x \notin \overline{(A \cup B)}$. Therefore $\overline{(A \cup B)} \subset \overline{A} \cup \overline{B}$.

Suppose that $x \in \overline{A} \cup \overline{B}$. Then every open neighborhood of x intersects A or B, so it intersects $A \cup B$. Thus $x \in \overline{A} \cup \overline{B}$, so $\overline{A} \cup \overline{B} \subset \overline{(A \cup B)}$. $\square$

Proposition 18. *Let $A, B \subset X$. If $A \subset B$, then $\overline{A} \subset \overline{B}$.*

Proof. Let $y \in \overline{A}$. Then every neighborhood of y intersects A. Since $A \subset B$, every neighborhood of y intersects B. Thus $y \in \overline{B}$. $\square$

Proposition 19. *Let $A \subset X$. Then*

(a) $A^\circ = (\overline{(A^c)})^c$;
(b) $\overline{A} = ((A^c)^\circ)^c$.

Proof. Exercise. $\square$

6.3. **Boundary Points.**

Definition 9. Let $A \subset X$. A *boundary point* of A is a point $z \in X$ such that every neighborhood of z intersects A and A^c. The *boundary* of A is the set of boundary points of A and is denoted ∂A.

Proposition 20. *Let $A \subset X$. Then*

- **(a)** $\partial A = \overline{A} \smallsetminus A^\circ$;
- **(b)** $\partial A = \overline{A} \cap \overline{A^c}$;
- **(c)** $\partial A = \partial A^c$;
- **(d)** $\overline{A} = A \cap \partial A$;
- **(e)** $A^\circ = A \smallsetminus \partial A$;
- **(f)** $\partial(\partial A) \subset \partial A$;
- **(g)** $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$.

Proposition 21. *Let $A \subset X$. Then $\partial A = \varnothing$ if and only if A is both open and closed.*

Proof.

($\Rightarrow$) Suppose that $\partial A = \varnothing$. Then $\overline{A} \subset A^\circ$. But $A^\circ \subset A \subset \overline{A}$, so $A^\circ = A = \overline{A}$. Thus A is both open and closed.

($\Leftarrow$) Suppose that A is both open and closed. Then $A^\circ = A = \overline{A}$, so $\partial A = \overline{A} \smallsetminus A^\circ = \varnothing$. $\square$

6.4. **Accumulation Points.**

Definition 10. Let $A \subset X$. A *accumulation point* of A is a point $z \in X$ such that every deleted neighborhood of z intersects A. The *derived set* of A is the set of accumulation points of A and is denoted A'.

Proposition 22. *Let $A, B \subset X$.*

- **(a)** $A \subset B \Rightarrow A' \subset B'$;
- **(b)** $(A \cup B)' = A' \cup B'$;
- **(c)** $\overline{A} = A \cup A'$.

Corollary 1. *A subset of X is closed if and only if it contains all of its accumulation points.*

6.5. **Isolated Points.**

Definition 11. Let $A \subset X$. An *isolated point* of A is a point $z \in A$ such that some deleted neighborhood of z is contained in A^c. The set of isolated points of A will be denoted $\dot{A}$.

Proposition 23. *Let $A \subset X$.*

- **(a)** $\dot{A} \subset A$;
- **(b)** $\dot{A} \subset \partial A$;
- **(c)** $\overline{A} = A' \sqcup \dot{A}$.

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